SECTION 8.6: INTEGRATION STRATEGIES

BASIC INTEGRATION FORMULAS:

- $\displaystyle\int du = \int 1\, du = u + C$
- $\displaystyle\int u^p\, du = \frac{1}{p+1}\, u^{p+1} + C, \quad p \neq -1$
- $\displaystyle\int \sin(u)\, du = -\cos(u) + C$
- $\displaystyle\int \cos(u)\, du = \sin(u) + C$
- $\displaystyle\int \csc(u)\cot(u)\, du = -\csc(u) + C$
- $\displaystyle\int \sec(u)\tan(u)\, du = \sec(u) + C$
- $\displaystyle\int \csc^2(u)\, du = -\cot(u) + C$
- $\displaystyle\int \sec^2(u)\, du = \tan(u) + C$
- $\displaystyle\int \frac{1}{u}\, du = \ln|u| + C$
- $\displaystyle\int e^u\, du = e^u + C$
- $\displaystyle\int \tan(u)\, du = \begin{cases} \ln|\sec(u)| + C \\ -\ln|\cos(u)| + C \end{cases}$
- $\displaystyle\int \cot(u)\, du = \begin{cases} -\ln|\csc(u)| + C \\ \ln|\sin(u)| + C \end{cases}$
- $\displaystyle\int \sec(u)\, du = \begin{cases} \ln|\sec(u) + \tan(u)| + C \\ -\ln|\sec(u) - \tan(u)| + C \end{cases}$
- $\displaystyle\int \csc(u)\, du = \begin{cases} -\ln|\csc(u) + \cot(u)| + C \\ \ln|\csc(u) - \cot(u)| + C \end{cases}$

If $a > 0$:

- $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \arcsin\left(\frac{u}{a}\right) + C = \sin^{-1}\left(\frac{u}{a}\right) + C$
- $\displaystyle\int \frac{1}{u^2 + a^2}\, du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$
- $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$

BASIC INTEGRATION PROPERTIES:

Suppose F and G are antiderivatives of f and g, respectively, on some open interval I.

- **CONSTANT MULTIPLE RULE:** $\displaystyle\int k \cdot f(x)\, dx = k\int f(x)\, dx = k \cdot F(x) + C$
- **SUM AND DIFFERENCE RULE:** $\displaystyle\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx = F(x) \pm G(x) + C$
- **LINEAR SUBSTITUTION:** If $a \neq 0$, then $\displaystyle\int f(ax+b)\, dx = \frac{1}{a}F(ax+b) + C$.

  For example, $\displaystyle\int \cos(x)\, dx = \sin(x) + C$, so $\displaystyle\int \cos(117x - 42)\, dx = \frac{1}{117}\sin(117x - 42) + C$.

SOME INTEGRATION STRATEGIES:

- Rewrite the integrand:
  - radicals as powers: $\sqrt{2x+1} = (2x+1)^{\frac{1}{2}}$, etc.
  - distributive properties: $A(B+C) = AB + AC$; $\dfrac{A+B}{C} = \dfrac{A}{C} + \dfrac{B}{C}$
  - complete the square: $x^2 + bx = \left(x + \dfrac{b}{2}\right)^2 - \dfrac{b^2}{4}$.
  - long division
- substitution: let u be what's . . .
  - in parentheses
  - in the denominator
  - in the exponent
- Integration by parts:
  - remember 'LIATE'
  - tabular
- Powers of trigonometric functions:
  - rewrite: $\sin^3(\theta) = (\sin(\theta))^3$, etc.
  - use trigonometric identities: $\cos^2(\theta) = 1 - \sin^2(\theta)$, $\dfrac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)$, etc.
  - 'keep back' factors for the du: if $u = \sin(\theta)$, then $du = \cos(\theta)\,d\theta$, etc.
- Trigonometric substitution:
  - does the integrand contain: $\sqrt{a^2 - u^2}$? Try $u = a\sin(\theta)$.
  - does the integrand contain: $\sqrt{u^2 + a^2}$? Try $u = a\tan(\theta)$.
  - does the integrand contain: $\sqrt{u^2 - a^2}$? Try $u = a\sec(\theta)$.
- Is the integrand a rational function with an easy-to-factor denominator? Try partial fractions.
- Try something and keep at it!

EXAMPLE: Find the following integrals. Check your answers!

1. $\displaystyle\int \arcsin(x)\,dx$

2. $\displaystyle\int x^2 \ln(x)\,dx$

3. $\displaystyle\int x \sin(x^2)\,dx$

4. $\displaystyle\int x^2 \sin(x)\,dx$

5. $\displaystyle\int x^2\sqrt{9-x^2}\,dx$

6. $\displaystyle\int x^2\sqrt{x^2-9}\,dx$

7. $\displaystyle\int \frac{x}{x^2-9}\,dx$

8. $\displaystyle\int \frac{1}{x^2-9}\,dx$

9. $\displaystyle\int e^{2x}\sqrt{1+e^{2x}}\,dx$

10. $\displaystyle\int e^{x}\sqrt{1+e^{2x}}\,dx$

11. $\displaystyle\int \sqrt{1+e^{2x}}\,dx$

12. $\displaystyle\int \frac{\sqrt{9+x^2}}{x}\,dx$

HOMEWORK: Section 8.6: 7 - 83 odd.

HINTS!

1. Integration by parts: let $u = \arcsin(x)$; $dv = dx$
2. Integration by parts: let $u = \ln(x)$; $dv = x^2\,dx$
3. u-substitution: let $u = x^2$
4. Tabular integration with $u = x^2$ and $dv = \sin(x)\,dx$
5. Trigonometric substitution: let $x = 3\sin(\theta)$
6. Trigonometric substitution: let $x = 3\sec(\theta)$
7. u-substitution: let $u = x^2 - 9$
8. Partial fractions.
9. u-substitution: let $u = 1 + e^{2x}$
10. u-substitution followed by trigonometric substitution: let $u = e^x$ and then let $u = \tan(\theta)$.
11. u-substitution: let $u = \sqrt{1 + e^{2x}}$
12. Trigonometric substitution: let $x = 3\tan(\theta)$

CALCULUS DIFFERENTIATION GUIDE

I. Basic Differentiation Rules:

Let f, and g be differentiable functions of x with c a constant.

1. $\dfrac{d}{dx}(c) = 0$
2. $\dfrac{d}{dx}(cf(x)) = cf'(x)$
3. $\dfrac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$
4. $\dfrac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$
5. $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$
6. $\dfrac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

II. Basic Differentiation Formulas:

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}u^n = nu^{n-1}\,u'$
2. $\dfrac{d}{dx}|u| = \dfrac{u}{|u|}u' = \dfrac{|u|}{u}\,u'$
3. $\dfrac{d}{dx}\sin(u) = \cos(u)\,u'$
4. $\dfrac{d}{dx}\cos(u) = -\sin(u)\,u'$
5. $\dfrac{d}{dx}\tan(u) = \sec^2(u)\,u'$
6. $\dfrac{d}{dx}\cot(u) = -\csc^2(u)\,u'$
7. $\dfrac{d}{dx}\sec(u) = \sec(u)\tan(u)\,u'$
8. $\dfrac{d}{dx}\csc(u) = -\csc(u)\cot(u)\,u'$
9. $\dfrac{d}{dx}\ln|u| = \dfrac{d}{dx}\ln(u) = \dfrac{1}{u}\,u'$
10. $\dfrac{d}{dx}\log_a|u| = \dfrac{d}{dx}\log_a(u) = \dfrac{1}{u\ln a}\,u'$
11. $\dfrac{d}{dx}e^u = e^u\,u'$
12. $\dfrac{d}{dx}a^u = a^u\,(\ln a)\,u'$
13. $\dfrac{d}{dx}\arcsin(u) = \dfrac{d}{dx}\sin^{-1}(u) = \dfrac{1}{\sqrt{1-u^2}}\,u'$
14. $\dfrac{d}{dx}\arccos(u) = \dfrac{d}{dx}\cos^{-1}(u) = -\dfrac{1}{\sqrt{1-u^2}}\,u'$
15. $\dfrac{d}{dx}\arctan(u) = \dfrac{d}{dx}\tan^{-1}(u) = \dfrac{1}{1+u^2}\,u'$
16. $\dfrac{d}{dx}\text{arccot}(u) = \dfrac{d}{dx}\cot^{-1}(u) = -\dfrac{1}{1+u^2}\,u'$
17. $\dfrac{d}{dx}\text{arcsec}(u) = \dfrac{d}{dx}\sec^{-1}(u) = \dfrac{1}{|u|\sqrt{u^2-1}}\,u'$
18. $\dfrac{d}{dx}\text{arccsc}(u) = \dfrac{d}{dx}\csc^{-1}(u) = -\dfrac{1}{|u|\sqrt{u^2-1}}\,u'$
19. $\dfrac{d}{dx}\sinh(u) = \cosh(u)\,u'$
20. $\dfrac{d}{dx}\cosh(u) = \sinh(u)\,u'$
21. $\dfrac{d}{dx}\tanh(u) = \text{sech}^2(u)\,u'$
22. $\dfrac{d}{dx}\coth(u) = -\text{csch}^2(u)\,u'$
23. $\dfrac{d}{dx}\text{sech}(u) = -\text{sech}(u)\tanh(u)\,u'$
24. $\dfrac{d}{dx}\text{csch}(u) = -\text{csch}(u)\coth(u)\,u'$

The following formulas are optional and are included for completeness:

1. $\dfrac{d}{dx}\text{arsinh}(u) = \dfrac{d}{dx}\sinh^{-1}(u) = \dfrac{1}{\sqrt{u^2+1}}\,u'$
2. $\dfrac{d}{dx}\text{arcosh}(u) = \dfrac{d}{dx}\cosh^{-1}(u) = \dfrac{1}{\sqrt{u^2-1}}\,u'$
3. $\dfrac{d}{dx}\text{artanh}(u) = \dfrac{d}{dx}\tanh^{-1}(u) = \dfrac{1}{1-u^2}\,u'$
4. $\dfrac{d}{dx}\text{arcoth}(u) = \dfrac{d}{dx}\coth^{-1}(u) = \dfrac{1}{1-u^2}\,u'$
5. $\dfrac{d}{dx}\text{arsech}(u) = \dfrac{d}{dx}\text{sech}^{-1}(u) = -\dfrac{1}{u\sqrt{1-u^2}}\,u'$
6. $\dfrac{d}{dx}\text{arcsch}(u) = \dfrac{d}{dx}\text{csch}^{-1}(u) = -\dfrac{1}{|u|\sqrt{1+u^2}}\,u'$

CALCULUS INTEGRATION GUIDE

I. Basic Integration Rules:

Let f and g be continuous functions of x with c a constant.

1. $\displaystyle\int cf(x)\,dx = c\int f(x)\,dx$
2. $\displaystyle\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\displaystyle\int f(g(x))\,g'(x)\,dx = \int f(u)\,du$, where we have substituted $u = g(x)$.

II. Basic Integration Formulas:

1. $\displaystyle\int u^n\,du = \frac{u^{n+1}}{n+1} + C,\ n \neq -1.$
2. $\displaystyle\int u^{-1}\,du = \int \frac{1}{u}\,du = \int \frac{du}{u} = \ln|u| + C$
3. $\displaystyle\int \sin(u)\,du = -\cos(u) + C$
4. $\displaystyle\int \cos(u)\,du = \sin(u) + C$
5. $\displaystyle\int \tan(u)\,du = \begin{cases} \ln|\sec(u)| + C \\ -\ln|\cos(u)| + C \end{cases}$
6. $\displaystyle\int \cot(u)\,du = \begin{cases} \ln|\sin(u)| + C \\ -\ln|\csc(u)| + C \end{cases}$
7. $\displaystyle\int \sec(u)\,du = \begin{cases} \ln|\sec(u) + \tan(u)| + C \\ -\ln|\sec(u) - \tan(u)| + C \end{cases}$
8. $\displaystyle\int \csc(u)\,du = \begin{cases} \ln|\csc(u) - \cot(u)| + C \\ -\ln|\csc(u) + \cot(u)| + C \end{cases}$
9. $\displaystyle\int \sec^2(u)\,du = \tan(u) + C$
10. $\displaystyle\int \csc^2(u)\,du = -\cot(u) + C$
11. $\displaystyle\int \sec(u)\tan(u)\,du = \sec(u) + C$
12. $\displaystyle\int \csc(u)\cot(u)\,du = -\csc(u) + C$
13. $\displaystyle\int e^u\,du = e^u + C$
14. $\displaystyle\int a^u\,du = \frac{1}{\ln a}\,a^u + C$

Let $a > 0$:

15. $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C$
16. $\displaystyle\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$
17. $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

III. Techniques:

1. Basic u-substitution: let u be the quantity in parentheses, the radicand, the denominator, or the exponent.
2. If the integrand is a rational function, divide if improper, otherwise split the numerator, resolve into partial fractions, or complete the square!
3. Integration by Parts: $\int u\,dv = uv - \int v\,du$. Rule of thumb: let $u =$ L(og.)I(nverse)A(lg.)T(rig.)E(xp.)
4. For trigonometric integrals:
   (a) Can you spare a cosine and convert the rest to sines?
   (b) Can you spare a sine and convert the rest to cosines?
   (c) Can you use a reduction formula?
      i. $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$.
      ii. $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$.
   (d) Can you spare a secant^2 and convert the rest to tangents ?
   (e) Can you spare a secant tangent and convert the rest to secants ?
   (f) If there are no secants, convert a tangent^2 to secant$^2 - 1$.
   (g) If you have an odd power of secant and no tangents, use parts by sparing a secant^2 for the dv.
   (h) To handle integrands with cosecants and cotangents, mimic the above strategies for secants and tangents.
   (i) If all else fails, try using a Pythagorean Conjugate, converting back to sines and cosines, or the infamous $z = \tan\left(\frac{\theta}{2}\right)$ substitution.
5. For integrands involving $\sqrt{a^2 - u^2}$, use the trig. sub. $u = a\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.
6. For integrands involving $\sqrt{u^2 - a^2}$, use the trig. sub. $u = a\sec\theta$, $0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$.
7. For integrands involving $\sqrt{u^2 + a^2}$, use the trig. sub. $u = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.